Chapter 14

14.1-1

$$2^{11} = 2048$$

$$\sum_{k=0}^{3} \binom{23}{k} = 2048$$

Therefore, this code satisfies the Hamming bound exactly for $t = 3$.

14.1-2

(a) There are $\binom{n}{j}$ ways in which j positions can be chosen from n. But for a ternary code, a digit can be mistaken for two other digits. Hence the number of possible errors in j places is

$$\binom{n}{j}(3-1)^j$$

or

$$\sum_{j=0}^{t} \binom{n}{j} 2^j \to 3^{n-k} \geq \sum_{j=0}^{t} \binom{n}{j} 2^j$$

(b) The $(11, 6)$ code for $t = 2$ satisfies the Hamming bound equality exactly.

14.1-3 For the $(18, 7)$ code to correct up to three errors, we have

$$2^{11} \geq \sum_{j=0}^{3} \binom{18}{j} = 988$$

Hence, there exists a possibility of a 3-error correcting, (18, 7) code. However, for $t = 4$, this inequality no longer holds; therefore, this code cannot correct all four error patterns.

14.2-1 Because any row $\mathbf{g}_i$ of $\mathbf{G}$ is a codeword, it satisfies

$$\mathbf{g}_i \mathbf{H}^H = 0$$

Therefore, $\mathbf{G}\mathbf{H}^T = 0$.

14.2-2 We have two codewords: For $d = 0$, the codeword is [0 0 0], and for $d = 1$ the codeword is [1 1 1]. The Hamming distance is 3. Thus, it can correct up to 1 error.

14.2-3 We have two codewords: For $d = 0$, the codeword is [0 0 0 0 0], and for $d = 1$ the codeword is [1 1 1 1 1]. The Hamming distance is 5. Thus, it can correct up to 2 errors.

14.2-4

(a) This is a systematic code.

(b)

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(c) The codewords are

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$$

(d) The minimum distance is 2; therefore this code cannot guarantee to correct even a single error.

14.2-5

(a)

$$\mathbf{G} = \begin{bmatrix} & 1 \\ & 1 \\ \mathbf{I}_k & \vdots \\ & 1 \end{bmatrix}_{k \times (k+1)}$$

(b) See Table S14.2-5b.

Table S14.2-5b

Data Word	Codeword
000	0000
001	0011
010	0101
011	0110
100	1001
101	1010
110	1100
111	1111

(c) This is a single-error detectable code.

(d)

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1 \times (k+1)}$$

Therefore, $\mathbf{cH}^T = 0$. If a single error occurs, $\mathbf{rH}^T = 1$, and if no error occurs, $\mathbf{rH}^T = 0$.

14.2-6 From Table S14.2-6, we can see that the distance between any two codewords is at least 3. Hence $d_{min} = 3$.

14.2-7 From Table S14.2-7, we can see that the distance between any two codewords is at least 3. Hence $d_{min} = 3$.

Table S14.2-6

Data Word	Codeword
000	000000
001	110001
010	111010
011	001011
100	011101
101	101100
110	100111
111	010110

Table 14.2-7

Data Word	Codeword
000	000000
001	001110
010	010101
011	011011
100	100011
101	101101
110	110110
111	111000

14.2-8

$$\mathbf{G} = \begin{bmatrix} & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 \\ \mathbf{I}_{11} & 1 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 \end{bmatrix}$$

We can see that when the data vector is 10111010101, the codeword is 101110101011110.

14.2-9

(a)

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) See Table S14.2-9b.

(c) The minimum distance is 3. Hence, this is a single-error correcting code. Since there are 6 possible single errors, all can be corrected. It can also correct one 2-error event.

(d) See Table S14.2-9d.

Table S14.2-9b

Data Word	Codeword
000	000000
001	001101
010	010110
011	011011
100	100111
101	101010
110	110001
111	111100

Table S14.2-9d

e	s
100000	111
010000	110
001000	101
000100	100
000010	010
000001	001
100100	011

(e) See Table 14.2-9e.

Table 14.2-9e

r	s	e	c	d
101100	110	010000	111100	111
001100	110	010000	010110	010
101010	000	000000	101010	101

14.2-10

(a) Solution is given in Problem 14.2-6.

(b) See Table S14.2-10b.

14.2-11

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

See Table S14.2-11.

14.2-12 We observe that the three 2-error patterns 100010, 010100, and 001001 have the same syndrome, namely, 111. Since the decoding table specifies $s = 111$ for $e = 100010$, whenever $e = 100010$ occurs, it will be corrected. The other two patterns will not be corrected. If, for example, $e = 010100$ occurs, $s = 111$; thus, we shall read from the decoding table $e = 100010$, and the error will not be corrected.

Table S14.2-10b

e	s
100000	111
010000	110
001000	011
000100	100
000010	010
000001	001
100010	101

Table S14.2-11

e	s
1000000	101
0100000	111
0010000	011
0001000	110
0000100	100
0000010	010
0000001	001

If we wish to correct the 2-error pattern 010100 (along with the six single-error patterns), the new decoding table is similar to that in Table 14.3 in the text, except for the last entry, which should be

e	s
010100	111

14.2-13 We should have

$$2^{n-k} \geq n+1 \text{ or } 2^{n-8} \geq n+1 \longrightarrow n-8 \geq \log_2(n+1)$$

This is satisfied for $n \geq 12$. Choose $n = 12$. This gives a $(12, 8)$ code. $\mathbf{H}^T$ is chosen to have 12 distinct rows of four elements with the last 4 rows forming an identity matrix. Hence,

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The number of nonzero syndromes= $16 - 1 = 15$. There are 12 single-error patterns. Hence, we may be able to correct 3 double-error patterns. The error patterns and their corresponding syndromes are shown in Table S14.2-13.

14.2-14

(a) See Table S14.2-14a.

This minimum distance between any two code words is $d_{\min} = 4$. Therefore, this code can correct all single-error patterns. Since the code over-satisfies the Hamming bound, it can also correct some 2-error and possibly some 3-error patterns. See Table S14.2-14.

Table S14.2-13

s				e											
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	0
1	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0
1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	1

Table S14.2-14a

Data word	Codeword
00	000000
01	011011
10	101110
11	110101

(b) See Table S14.2-14b.

14.3-1

(a) See Table S14.3-1a.

(b) From Table S14.3-1a, it can be seen that the minimum distance between any two codes is 3. Hence, this is a single-error-correcting code.

(c) See Table S14.3-1c.

(d) The received data is 1101100, $r(x) = x^6 + x^5 + x^3 + x^2$. When $r(x)$ is divided by $g(x)$, its remainder is $x^2 + 1$. Then $e = 1000000$, therefore, $c = r \oplus e = 0101100$. Hence, $d = 0101$.

14.3-2

$$g(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1$$

$$d_1(x) = x^7 + x^6 + x^5 + x^4$$

and

$$d_2(x) = x^{11} + x^9 + x^7 + x^5 + x^3 + x$$

Table S14.2-14b

e	s
100000	1110
010000	1011
001000	1000
000100	0100
000010	0010
000001	0001
110000	0101
101000	0110
100100	1010
100010	1100
011000	0011
010010	1001
000111	0111
001101	1101

Table S14.3-1a

d	c
1111	1111111
1110	1110100
1101	1101001
1100	1100010
1011	1011000
1010	1010011
1001	1001110
1000	1000101
0111	0111010
0110	0110001
0101	0101100
0100	0100111
0011	0011101
0010	0010110
0001	0001011
0000	0000000

Table S14.3-1c

e	s
1000000	101
0100000	111
0010000	011
0001000	110
0000100	100
0000010	010
0000001	001

$$c_1(x) = d_1(x)g(x) = x^{18} + x^{17} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 + x^4$$

Thus,

$$c_1 = 0000110001110111001 0000$$

$$c_2(x) = d_2(x)g(x) = x^{22} + x^{18} + x^{17} + x^{15} + x^{13} + x^8 + x^5 + x^4 + x^3 + x^2 + x$$

so

$$c_2 = 10001101010000100111110$$

14.3-3 $x^5 + x^4 + x^2 + 1 = (x+1)(x^3+1) = (x+1)(x+1)(x+1) = (x+1)^3$

14.3-4 Note that $x^5 + x^4 + x^2 + 1 = (x+1)(x^4+x+1)$.

Dividing x^4+x+1 by $x+1$, we get a remainder of 1. Hence, $(x+1)$ is not factor of (x^4+x+1). The second-order prime factors not divisible by $x+1$ are x^2 and x^2+x+1. Since (x^4+x+1) is not divisible by x^2, we try dividing by (x^2+x+1). This also yields a remainder of 1. Hence, x^4+x+1 does not have either a first- or a second-order factor. This means that it cannot have a third-order factor either. Hence, $x^5+x^4+x^2+1 = (x+1)(x^4+x+1)$.

14.3-5 $x^7 + 1 = (x+1)(x^6+x^5+x^4+x^3+x^2+x+1)$.
First try dividing $(x^6+x^5+x^4+x^3+x^2+x+1)$ by $(x+1)$. It does not divide. Then try dividing by (x^2+1). It does not divide. Next try dividing by (x^3+1). It does not divide, either. Now try dividing by (x^3+x+1). It divides. We find

$$x^7+1 = (x+1)(x^2+x+1)(x^3+x^2+1)$$

14.3-6 For a single-error-correcting $(7,4)$ cyclic code with a generator polynomial

$$g(x) = x^3 + x^2 + 1$$

$k = 4, n = 7$

$$\begin{bmatrix} x^3 g(x) \\ x^2 g(x) \\ x g(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} x^6 + x^5 + x^3 \\ x^5 + x^4 + x^2 \\ x^4 + x^3 + x \\ x^3 + x^2 + 1 \end{bmatrix}$$

Hence,

$$\mathbf{G}' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Each code word is found by matrix multiplication $\mathbf{c} = \mathbf{dG}'$

$$c = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The remaining codes are found in a similar manner. See Table S14.3-6 below.

Table S14.3-6

d	c
1111	1001011
1110	1000110
1101	1010001
1100	1011100
1011	1111111
1010	1110010
1001	1100101
1000	1101000
0111	0100001
0110	0101110
0101	0111001
0100	0110100
0011	0010111
0010	0011010
0001	0001101
0000	0000000

14.3-7

$$g(x) = x^3 + x^2 + 1$$

The desired form is

$$\mathbf{G} = \begin{bmatrix} & h_{11} & h_{21} & \cdots & h_{m1} \\ & h_{12} & h_{22} & \cdots & h_{m2} \\ \mathbf{I}_k & \vdots & \vdots & \ddots & \vdots \\ & h_{1k} & h_{2k} & \cdots & h_{mk} \end{bmatrix}$$

The codewords are found by using $\mathbf{c} = \mathbf{d} \cdot \mathbf{G}$. Proceeding with matrix multiplication and noting that

$$0+0=0,\ 0+1=1+0=1,\ 1+1=0 \text{ and} 0 \times 0 = 0, 0 \times 1 = 1 \times 0 = 0, 1 \times 1 = 1$$

we get codewords

$$c_{15} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$c_{14} = [1110]G = [1110010]$$

and so on. See Table S14.3-7 for a full list of codewords.

14.3-8

(a)

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Table S14.3-7

d	c
1111	1111111
1110	1110010
1101	1101000
1100	1100101
1011	1011100
1010	1010001
1001	1001011
1000	1000110
0111	0111001
0110	0110100
0101	0101110
0100	0100011
0011	0011010
0010	0010111
0001	0001101
0000	0000000

(b) The code is found by matrix multiplication: $\mathbf{c} = \mathbf{d} \cdot \mathbf{G}$.
In general, $g(x) = g_1 x^{n-k} + g_2 x^{n-k-1} + \ldots + g_{n-k+1}$. For this case, $g_1 = 1$, $g_2 = 1$, $g_3 = 0$, and $g_4 = 1$. We can also have the $\mathbf{G}$ matrix in the following systematic form

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

This the desired form.

Table S14.3-8

d	c
1111	1101001
1110	1100010
1101	1111111
1100	1110100
1011	1000101
1010	1001110
1001	1010011
1000	1011000
0111	0110001
0110	0111010
0101	0100111
0100	0101100
0011	0011101
0010	0010110
0001	0001011
0000	0000000

(c) See Table S14.3-8. All codewords are separated by a minimum distance of 3 bits. Hence, this is a single-error correcting code.

14.3-9 $g(x) = x^3 + x + 1$

$$\mathbf{G}' = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and by applying Gauss elimination, we have

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

14.3-10 The code can correct any 3 bursts of length 10 or less. It can also correct any 3 random errors in each code word.

14.3-11

(a) $g(x) = x^{10} + x^8 + x^5 + x^4 + x^2 + x + 1$. Its generator matrix is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) The corresponding codeword is

100110111000010

(c) $n = 15, k = 5$, therefore $m = 4$. $n - k = 10 \leq 4 \times 3$. Hence, $t = 3$. This code can correct up to 3 errors.

14.4-1

$$P_{eu} = kQ(\sqrt{2E_b/\mathcal{N}}) = 12Q(\sqrt{2 \times 9}) = 1.3254 \times 10^{-4}$$

$$P_{ec} = (\frac{23}{4}) \left[Q(\sqrt{\frac{2kE_b}{n\mathcal{N}}})\right]^4 = (\frac{23}{4}) \left[Q(\sqrt{9.3913})\right]^4 = 8.1192 \times 10^{-12}$$

To achieve a value 8.1192×10^{-12} for P_{eu}, we need a new value of SNR. Then $8.1192 \times 10^{-12} = 12Q(\sqrt{2\text{SNR}})$, SNR $= 25.12$. It means the increase of SNR from 9 to 25.12.

14.4-2 The burst (of length 5) detection ability is obvious. The single error correcting ability can be demonstrated as follows. If in any segment of b digits a single error occurs, it will violate the parity in that segment. Hence, we locate the segment where the error exists. This error will also cause a parity violation in the augmented segment. By checking which bit in the augmented segment violates the parity, we can locate exactly the position of the wrong bit and can correct it.

14.5-1 The received bits are **01 00 01 00 10 11 11 00**. The decoded information sequence is **11010010**.